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ABSTRACT

We study the boundedness and asymptotic properties of solutions as $t \to \infty$ of the nonlinear Volterra equation

$$\text{(V)} \qquad u(t) + (b*Au)(t) \ni f(t) \qquad (0 \leq t < \infty) ,$$

where $b : [0,\infty) \to R$ is a given kernel, A is a maximal monotone (possibly multi-valued) operator on a real Hilbert space H, $*$ is the convolution, and $f : [0,\infty) \to H$ is a given function. The special case $A = \partial\varphi$, where $\varphi : H \to (-\infty,+\infty]$ is a proper, convex, l.s.c. function is also considered. The problem of existence and uniqueness of solutions has been studied previously. Two principal types of results are derived; sufficient conditions are obtained on the kernel b, the operator A, and the forcing term f such that either (i) $u \in L^\infty(0,\infty;H)$ and $u \to 0$ as $t \to \infty$ strongly in H, or (ii) $u \in L^\infty(0,\infty;H)$ and $u \to u_\infty$ as $t \to \infty$ strongly in H, where u_∞ is the unique solution of an appropriate limiting equation associated with (V). The results are established by obtaining a priori estimates by combining energy methods with frequency domain techniques. The results are natural generalizations for the abstract equation (V) of comparable known results for the scalar equation in which A is a real function. Of several applications discussed the principal one is an analysis of the asymptotic behaviour of solutions of the physically interesting problem of heat flow in a material with memory.

AMS (MOS) Subject Classifications: 45D05, 45K05, 45G99, 45M05, 45M10, 47H05, 47H15.

Key Words: Nonlinear Volterra equation, Maximum monotone operators on a Hilbert space, Subdifferential of a proper, convex, l.s.c. function, Boundedness, Asymptotic behaviour, Limiting equation, Strong solutions, Generalized solutions, Energy methods, Frequency domain methods, Heat flow, Materials with memory

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SIGNIFICANCE AND EXPLANATION

Consider nonlinear heat flow in a homogeneous bar of unit length of a material with memory with the ends of the rod maintained at zero temperature and with the history of temperature prescribed for time $t \leq 0$. For such materials the internal energy and heat flux are functionals (rather than functions) of the temperature and of the gradient of temperature respectively. Application of the law of balance of heat leads to a nonlinear Volterra integrodifferential equation, together with appropriate boundary and initial conditions, which model the physical problem. This initial boundary value problem, which cannot be solved explicitly and which is difficult to analyse, can be transformed by standard methods to the general equation (V) given in the Abstract. The resulting kernel b can be expressed in terms of the internal energy and heat flux relaxation functions. These are presumed to be known for the physical problem. The operator A in (V) is a nonlinear differential operator together with boundary conditions, and the forcing term f in (V) depends on the given initial temperature distribution, the given external heat supply, and the given history of temperature.

The purpose of this paper is to develop a general theory which gives sufficient conditions in terms of the kernel b, the operator A, and the forcing term f for the solution u of (V) to be bounded on $0 \leq t < \infty$ and which further assures that the solution u tends to a limit u_∞ as $t \to \infty$; under certain conditions $u_\infty = 0$, under others u_∞ is the unique solution of an appropriate "limit equation" associated with (V). As one special case of this theory we give a complete analysis of the boundedness and asymptotic properties of the solution of the above heat flow problem, under physically reasonable assumptions concerning the relaxation functions, the nonlinear operator, the initial temperature distribution, and the external heat supply.

ASYMPTOTIC PROPERTIES OF SOLUTIONS OF NONLINEAR ABSTRACT VOLTERRA EQUATIONS

Ph. Clément*, R. C. Mac Camy**, and J. A. Nohel***

1. Introduction and Preliminaries. The purpose of this paper is to discuss the boundedness and asymptotic behaviour as $t \to \infty$ of solutions of the nonlinear Volterra equation

(V) $$u(t) + (b*Au)(t) \ni f(t) \quad (0 \leq t < \infty) .$$

The setting for (V) is $b : [0,\infty) \to R$ is a given kernel, A is a (possibly multivalued) maximal monotone operator on a real Hilbert space H, $f : [0,\infty) \to H$ is a given function, and $*$ denotes the convolution: $(\alpha*g)(t) = \int_0^t \alpha(t-\tau)g(\tau)d\tau$. The integral in (V) is taken in the sense of Bochner.

The following general assumptions will be assumed throughout:

(H_b) $$b(t) = b_\infty + B(t),\ b(0) > 0,\ b_\infty \geq 0,\ B,B' \in L^1(0,\infty) ;$$

(H_m) A maximal monotone on H ;

(H_f) $$f(t) = f_\infty + F(t),\ F \in W^{1,2}_{loc}([0,\infty;H)),\ F' \in L^2(0,\infty;H),\ f_\infty \in H ;$$

here $' = d/dt$, H is a real Hilbert space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$, and W denotes the usual Sobolev space. The special case of (H_m):

(H_φ) $$\begin{cases} A = \partial\varphi, \text{ where the function } \varphi : H \to (-\infty,\infty] \text{ is convex,} \\ \text{lower semicontinuous, and proper} \end{cases}$$

will also play an important role in the theory. For definitions and standard results

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concerning maximal monotone operators and the special case of a subdifferential the reader is referred to Brézis [3].

We shall first comment briefly on the existence and uniqueness of solutions of (V) for which boundedness and asymptotic properties will be studied. A special case of a general result of Crandall and Nohel [5; Theorem 4] (see also Gripenberg [7]) is:

Proposition 1.1. *If* (H_b) *with* $B, B' \in L^1_{loc}(0,\infty)$, (H_m), (H_f) *with* $F' \in L^2_{loc}(0,\infty;H)$ *are satisfied, and if* $B' \in BV_{loc}[0,\infty)$, $f(0) \in \overline{D(A)}$, *then* (V) *has a unique generalized solution* $u \in C([0,\infty); \overline{D(A)})$. *If, in addition* $F' \in BV_{loc}([0,\infty);H)$ *and* $f(0) \in D(A)$, *then* $u \in W^{1,2}_{loc}([0,\infty);H)$ *and* u *is a strong solution of* (V) *on* $[0,\infty)$.

In the special case of a subdifferential Proposition 1.1 can be strengthened to (see Crandall and Nohel [5, Section 4], also Londen [10]):

Proposition 1.2. *Let* b, f *satisfy the assumptions of Proposition* 1.1 *and let* (H_φ) *hold. If* $f(0) \in \overline{D(\varphi)}$, *then* (V) *has a unique strong solution on* $[0,\infty)$ *such that* $\sqrt{t}\, u' \in L^2_{loc}([0,\infty);H)$; *if* $f(0) \in D(\varphi)$, *then* $u' \in L^2_{loc}([0,\infty);H)$.

Remark 1.3. For convenience of the reader we recall the notion of generalized solution (see [5]). Let $\lambda > 0$ and let $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$, $J_\lambda = (I + \lambda A)^{-1}$ be the Yosida approximation of A. Noting that $A_\lambda : H \to H$ is Lipschitz continuous (with Lipschitz constant $\frac{1}{\lambda}$), a simple contraction mapping argument shows that the approximating equation

$$(V_\lambda) \qquad u_\lambda + b*A_\lambda u_\lambda = f$$

has a unique strong solution u_λ on $[0,T]$ for every $T > 0$, under the assumptions of Propositions 1.1 or 1.2. Moreover, under these assumptions u_λ is also a strong solution of the differentiated approximating equation

$$(V'_\lambda) \qquad \frac{du_\lambda}{dt} + b(0)A_\lambda u_\lambda + B'*A_\lambda u_\lambda = F', \quad u(0) = f(0) ,$$

on $[0,T]$ for every $T > 0$. What is established in [5] is that $\lim_{\lambda \to 0^+} u_\lambda = u$ in $C([0,T];H)$ exists, regardless of whether or not (V) has a strong solution, and this function u solves (V) in a sense made precise in [5]. For this reason this function u is called a generalized solution of (V).

The theorems in Sections 2 and 3 are valid for both strong and generalized solutions of (V); while the proofs are carried out for strong solutions, they may be established for

generalized solutions by first obtaining the estimates (independent of λ and t) from the approximating equation (V'_λ) and then letting $\lambda \downarrow 0^+$.

There are two approaches for obtaining the a priori estimates needed for our results. The more direct one is an energy method in which the estimates needed are obtained directly from the equivalent differentiated form of (V):

$$(V') \qquad \frac{du}{dt} + b(0)Au + B'*Au \ni F' \quad (0 < t < \infty) ,$$

$u(0) = f(0)$. This method, discussed in Section 3 under the essential assumption (H_φ) $(A = \partial\varphi)$, consists of taking the scalar product of (V') by any element $v \in \partial\varphi(u)$ and integrating over an arbitrary interval $(0,T)$ to obtain the estimates under suitable assumptions concerning b, φ, and f. This technique is an extension of a similar approach which can be used to study evolution equations (take $b \equiv 1$ in (V)) in the case $A = \partial\varphi$. The crucial point is an appropriate condition on b (see the frequency domain condition (F) in Lemma 2.2) and appropriate growth conditions on the function φ to deduce $\frac{du}{dt} \in L^2(0,\infty;H)$. Our motivation for studying this approach is the physical problem of heat flow in materials with memory to which these results are applied in Section 4, and where $b_\infty > 0$ in assumption (H_b).

The approach in the more general case of A satisfying (H_m) is different; there appears to be no direct way to establish $\frac{du}{dt} \in L^2(0,\infty;H)$ when A is not a subdifferential. Instead (V') is transformed to a different equivalent form (equation (2.1) below) using the resolvent kernel associated with B'. An energy technique consisting of taking the scalar product of the transformed equation by u and by $\sqrt{t}\, u$ and integrating from 0 to T, $T > 0$ arbitrary, is used to deduce the estimates which imply $u, \sqrt{t}\, u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$ under appropriate assumptions. This method is explored in Section 2. An example of the equation (V) with A a maximal monotone operator which is not a subdifferential and to which the theory of Section 2 can be applied, is also discussed. Corollaries 2.4, 2.5, and Theorem 2.6 may be viewed as natural generalizations to Hilbert space of results of Levin [8] and Londen [9] which describe the limiting behaviour as $t \to \infty$ of solutions of (V) in the scalar case in which the operator A is a real function.

While not directly related to the present study it should also be noted that positivity of solutions of (V) and their asymptotic properties, have recently been studied by different methods by Clément and Nohel [4], in the setting of A an m-accretive operator on a real Banach space X.

2. Boundedness and Asymptotic Properties when A is Maximal Monotone. Let the general assumptions (H_b), (H_m), (H_f) be satisfied and let u be a solution of (V) on $[0,\infty)$. It follows that (V) is equivalent to the Cauchy problem

$$\text{(V')} \qquad \frac{du}{dt} + b(0)Au + B'*Au \ni F' \quad (0 < t < \infty), \quad u(0) = f(0) .$$

Let k be the resolvent kernel associated with B', defined to be the unique solution of the linear Volterra equation

$$\text{(k)} \qquad b(0)k(t) + (B'*k)(t) = -\frac{B'(t)}{b(0)} \quad \text{a.e. for } 0 \le t < \infty ;$$

by standard results assumption (H_b) implies that $k \in L^1_{loc}(0,\infty)$.

Regarding (V') as a "linear" equation for Au, the variation of constants formula for Volterra equations [13] and an integration by parts shows that (V') (and hence also (V)) is equivalent to the Cauchy problem

$$(2.1) \qquad \frac{1}{b(0)}\frac{du}{dt} + \frac{d}{dt}(k*u) + Au \ni f_1 \quad (0 < t < \infty), \quad u(0) = f(0) ,$$

where $f_1 : [0,\infty) \to H$ is the function given by either

$$(2.2) \qquad f_1(t) = \frac{1}{b(0)} F'(t) + f(0)k(t) + (k*F')(t) \quad (0 \le t < \infty)$$

or

$$(2.3) \qquad f_1(t) = \frac{1}{b(0)} F'(t) + k(0)f(t) + (k'*f)(t) \quad (0 \le t < \infty) .$$

We shall use an energy method based on taking the scalar product of (2.1) by u, and also by $\sqrt{t}\,u$, and obtain a priori estimates by integrating over an arbitrary interval $[0,T]$. We will first state the general result for (2.1) and then interpret it for (V).

Theorem 2.1. Let u be a strong or generalized solution of the Cauchy problem (2.1) on $(0,\infty)$. Let $T > 0$ be given and let there exist constants $\varepsilon, \eta \in R$ such that

$$(2.4) \qquad \text{if } v \in Au, \text{ then } \int_0^T (v,u)dt \ge \varepsilon \int_0^T |u|^2 dt \; (u \in D(A)) ,$$

$$(2.5) \qquad \text{for every } w \in L^2(0,T;H) \int_0^T \left(w(t), \frac{d}{dt}(k*w)(t)\right)dt \ge \eta \int_0^T |w|^2 dt ,$$

$$(2.6) \qquad \varepsilon + \eta > 0 .$$

(a) If $f_1 \in L^2(0,\infty;H)$, then $u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$;

(b) If also $\sqrt{t}\, k' \in L^1(0,\infty)$ and $\sqrt{t}\, f_1 \in L^2(0,\infty;H)$, then $\sqrt{t}\, u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$. Consequently, $|u(t)| = O(\frac{1}{\sqrt{t}})$ as $t \to \infty$ and $u(t) \to 0$ strongly as $t \to \infty$.

We remark that no claim is made that the rate $|u(t)| = O(\frac{1}{\sqrt{t}})$ as $t \to \infty$ is optimal.

The coercivity assumption (2.4) concerning the maximal monotone operator A is natural for the problem in light of comparable assumptions in evolution equations. Assumption (2.5) and the hypotheses concerning k, k' will be justified in Lemmas 2.2, 2.3, below. Two different classes of kernels b in (V) are considered, each of which lead to the energy inequality (2.5), the first with $\eta = 0$, the second with $\eta > 0$, and for each of which $\sqrt{t}\, k' \in L^1(0,\infty)$. These lemmas, together with appropriate assumptions on the forcing function f in (V), permit an easy interpretation of Theorem 2.1 for solutions of (V). This will be done in Corollaries 2.4 and 2.5 below. The proof of Lemma 2.2 appears in Appendix 1. Lemma 2.3 is an extension of a result of Mac Camy [12] which in its present form was recently established by M. Tangredi [18].

Lemma 2.2. (a) Let b satisfy assumption (H_b) with $b_\infty > 0$, and let b satisfy the frequency domain condition

(F) there exists $\delta > 0$ such that $b_\infty + \operatorname{Inf}_{\eta \in R} [-\eta \operatorname{Im} \hat{B}(i\eta)] \geq \delta$,

where $\hat{B}(i\eta) = \int_0^\infty \exp(-i\eta t) B(t)dt$. Then the resolvent kernel k of B' satisfies $k \in L^1(0,\infty)$.

(b) If also $B' \in L^2(0,\infty)$, then $k \in L^2(0,\infty)$; if also $B'' \in L^1(0,\infty)$, then $k' \in L^1(0,\infty)$.

(c) If the assumptions of (a) are satisfied, $B'' \in L^1(0,\infty)$, and B is a kernel of positive type on $[0,\infty)$, then for every $T > 0$ and for every $w \in L^2(0,T)$

$$\int_0^T w(t) \frac{d}{dt} (k*w)(t)dt \geq 0 .$$

(d) If the assumptions of (a) and (b) are satisfied, and $\sqrt{t}\, B' \in L^1(0,\infty) \cap L^2(0,\infty)$, $\sqrt{t}\, B'' \in L^1(0,\infty)$, then $\sqrt{t}\, k \in L^1(0,\infty) \cap L^2(0,\infty)$, and $\sqrt{t}\, k' \in L^1(0,\infty)$.

Lemma 2.3. Let b satisfy assumption (H_b) with $b_\infty = 0$, and let

(i) $t^j B^{(m)} \in L^1(0,\infty)$ $(j = 0,1,2;\ m = 0,1,2)$, $t^3 B \in L^1(0,\infty)$,

(ii) B be strongly positive on $[0,\infty)$.

Let k be the resolvent kernel of B'. Then:

(a) $k \in C^1[0,\infty)$;

(b) $k(t) = k_\infty + K(t)$, $k_\infty = (\int_0^\infty B(t)dt)^{-1} > 0$, $K^{(m)} \in L^1(0,\infty)$ $(m = 0,1,2)$;

(c) if also $B, B', \sqrt{t}\, B, \sqrt{t}\, B' \in L^2(0,\infty)$ one has $K, \sqrt{t}\, K \in L^2(0,\infty)$;

(d) for every $T > 0$ and for every $w \in L^2(0,T)$ there exists $\eta > 0$ such that

$$\int_0^T w(t) \frac{d}{dt} (k*w)(t)dt \geq \eta \int_0^T |w(t)|^2 dt ;$$

(e) if assumptions (i) hold for $j,m = 0,1,2,3$, $t^4 B \in L^1(0,\infty)$, and assumptions (ii) hold, one has $\sqrt{t}\, k' \in L^1(0,\infty)$.

We shall mention some examples of kernels b which satisfy the assumptions of Lemmas 2.2 and 2.3.

Let

(2.7) $B : [0,\infty) \to \mathbf{R}^+$ be positive, nonincreasing, and convex

and satisfy the smoothness and integrability assumptions in (H_b). Then B is a kernel of positive type on $[0,\infty)$ (see [14]), and

$$-\eta \operatorname{Im}\hat{B}(i\eta) = \eta \int_0^\infty \sin\eta t\, B(t)dt \geq 0 \quad (\eta \in \mathbf{R}) .$$

Thus if $b_\infty > 0$ is any constant, $b(t) = b_\infty + B(t)$ satisfies the frequency domain condition (F) with $\delta = b_\infty$, and (see Lemma 2.2(a)) $k \in L^1(0,\infty)$. If, in addition, B satisfies the remaining smoothness and integrability assumptions of Lemma 2.2, all conclusions of Lemma 2.2, and assumptions (2.5) with $\eta = 0$, and $\sqrt{t}\, k' \in L^1(0,\infty)$ of Theorem 2.1 are satisfied.

Consider again B in (2.7). In addition, assume that

(2.8) the measure dB' has a nonzero absolutely continuous part;

then (see [14, Corollary 2.2]), B is strongly positive on $[0,\infty)$ (for example, $B \in C[0,\infty)$, $(-1)^k B^{(k)}(t) \geq 0$, $0 < t < \infty$, $k = 0,1,2$, $B'(t) \not\equiv 0$). Thus if B satisfies (2.7), (2.8), the integrability and smoothness assumptions of Lemma 2.3, and if $b(t) \equiv B(t)$ $(b_\infty = 0)$, then all conclusions of Lemma 2.3, assumptions (2.5) with [illegible] and $\sqrt{t}\, k' \in L^1(0,\infty)$ of Theorem 2.1 are satisfied.

Next, consider

$$(2.9) \qquad B(t) = \sum_{j=1}^{m} B_j e^{-\lambda_j t} \cos \omega_j t \quad (B_j \geq 0,\ \lambda_j \geq 0,\ \omega_j \in \mathbf{R})$$

with strict inequalities holding for at least one j (if $\omega_j = 0$, $j = 1,\dots,m$, B satisfies both (2.7), (2.8)). This function B is strongly positive on $[0,\infty)$ (see [14]), since by direct calculation

$$\operatorname{Re} \hat{B}(i\eta) = \frac{1}{2} \sum_{j=1}^{m} B_j \lambda_j \left(\frac{1}{\lambda_j^2 + (\eta - \omega_j)^2} + \frac{1}{\lambda_j^2 + (\eta + \omega_j)^2} \right) \quad (\eta \in \mathbf{R}) .$$

Moreover, B satisfies all other assumptions of Lemma 2.3. Thus if $b(t) = B(t)$ $(b_\infty = 0)$, all conclusions of Lemma 2.3, assumptions (2.5) with $\eta > 0$ and $\sqrt{t}\, k' \in L^1(0,\infty)$ of Theorem 2.1 are satisfied.

For the kernel B in (2.9) one has

$$-\eta \operatorname{Im} \hat{B}(i\eta) = \sum_{j=1}^{m} B_j \frac{\eta^2(\eta^2 + \lambda_j^2 - \omega_j^2)}{(\lambda_j^2 + w_j^2 - \eta^2)^2 + 4\lambda_j^2 \eta^2} \quad (\eta \in \mathbf{R}) .$$

Thus $b(t) = b_\infty + B(t)$, where $b_\infty > 0$ is any constant, satisfies the frequency domain condition (F) of Lemma 2.2 if $\lambda_j \geq \omega_j$ $(j = 1,\dots,m)$. Evidently, b is a kernel of positive type on $[0,\infty)$. Therefore, if $b(t) = b_\infty + B(t)$, $b_\infty > 0$, B defined by (2.9) with $\lambda_j \geq \omega_j$ $(j = 1,\dots,m)$, all conclusions of Lemma 2.2 (but not of Lemma 2.3), assumptions (2.5) with $\eta = 0$ and $\sqrt{t}\, k' \in L^1(0,\infty)$ of Theorem 2.1 are satisfied.

Incidentally, if $b_\infty > 0$ is any constant, and if

$$B(t) = \sum_{i=1}^{m} B_j e^{-\lambda_j t} \sin \omega_j t \quad (\lambda_j \geq 0,\ \omega_j \geq 0) \tag{2.10}$$

with strict inequalities holding for at least one j, then the frequency domain condition (F) of Lemma 2.2 is satisfied with $\delta = b_\infty$. However, such a kernel b is not of positive type.

Lemmas 2.2 combined with appropriate assumptions on A and f yield the following easy interpretation of Theorem 2.1 for solutions of (V).

Corollary 2.4. *Let assumptions* (H_b) *with* $b_\infty > 0$, (H_m) *and* (H_f) *with* f_∞ *arbitrary be satisfied. In addition, assume that* b *satisfies the hypotheses of Lemma* 2.2, *and that* $\sqrt{t}\, F' \in L^2(0,\infty;H)$. *Let* u *be a strong or generalized solution of* (V) *on* $[0,\infty)$. *If the coercivity assumption* (2.4) *holds with* $\varepsilon > 0$, *then* u *and* $\sqrt{t}\, u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$ *and* $u(t) \to 0$ *strongly as* $t \to \infty$.

Indeed, define f_1 by (2.2). By (H_f) and Lemma 2.2 $(k \in L^1(0,\infty) \cap L^2(0,\infty)$, $k' \in L^1(0,\infty))$ one trivially has $f_1 \in L^2(0,\infty;H)$. By Lemma 2.2 one also has $\sqrt{t}\, k \in L^1(0,\infty) \cap L^2(0,\infty)$ and $\sqrt{t}\, k' \in L^1(0,\infty)$. These together with the assumption $\sqrt{t}\, F' \in L^2(0,\infty;H)$ used in (2.2) show that $\sqrt{t}\, f_1 \in L^2(0,\infty;H)$; the fact that $\sqrt{t}\,(k*F') \in L^2(0,\infty;H)$ in (2.2) follows from the straightforward estimate

$$\int_0^T t\left|\int_0^t k(t-s)F'(s)ds\right|^2 dt \leq 2\|k\|^2_{L^1(0,\infty)} \|\sqrt{t}\, F'\|^2_{L^2(0,\infty;H)} + 2\|\sqrt{t}\, k\|^2_{L^1(0,\infty)} \|F'\|^2_{L^2(0,\infty;H)} \quad (\forall T > 0)\ .$$

By Lemma 2.2 again, (2.5) holds with $\eta = 0$. Thus if $\varepsilon > 0$ in (2.4), the result of Corollary 2.4 follows by applying Theorem 2.1.

Lemma 2.3 combined with appropriate assumptions on A and f yield a different interpretation of Theorem 2.1 for solutions of (V).

Corollary 2.5. Let assumptions (H_b) with $b_\infty = 0$, (H_m) and (H_f) with $f_\infty = 0$ be satisfied. In addition, assume that $b(t) = B(t)$ satisfies the assumptions of Lemma 2.3, and that $f = F$ also satisfies $f, \sqrt{t}\, f, \sqrt{t}\, f' \in L^2(0,\infty;H)$. Let u be a strong generalized solution of (V) on $[0,\infty)$. If the operator A satisfies the coercivity assumption (2.4) with $\varepsilon \geq 0$ (or even $\varepsilon > -\eta$, where $\eta > 0$ is the constant in Lemma 2.3d), then u and $\sqrt{t}\, u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$, and $u(t) \to 0$ strongly as $t \to \infty$.

The proof of Corollary 2.5 is similar to that of Corollary 2.4, except that f_1 must now be defined by (2.3), and Lemma 2.3 is used in place of Lemma 2.2. Note also that the additional assumptions concerning $f, \sqrt{t}\, f$ are essential.

The important case $b_\infty = 0$ in (H_b), $b \equiv B$ satisfying the assumptions of Lemma 2.3, and $f_\infty \neq 0$ in (H_f) is not covered by Corollary 2.5. In this situation Theorem 2.1 must be modified in the following manner.

Theorem 2.6. Let the assumptions (H_b) $(b_\infty = 0)$, (H_m), (H_f) with f_∞ arbitrary, and the assumptions of Lemma 2.3 be satisfied. In addition, assume that $F, \sqrt{t}\, F, \sqrt{t}\, f' \in L^2(0,\infty;H)$. Let u be a strong or generalized solution of (V) on $[0,\infty)$, let u_∞ be the unique solution of the limit equation corresponding to (V):

$$(V_\infty) \qquad u_\infty + \left(\int_0^\infty B(t)dt\right)Au_\infty \ni f_\infty .$$

Let the operator A satisfy the coercivity condition:

$$(2.11) \qquad \begin{cases} \text{if } v \in Au \text{ and } v_\infty \in Au_\infty \text{ and } T > 0, \text{ then} \\ \int_0^T (v(t) - v_\infty, u(t) - u_\infty)dt \geq \varepsilon \int_0^T |u(t) - u_\infty|^2 dt \\ \text{for some } \varepsilon \geq 0 \ (\varepsilon > -\eta \text{ is sufficient; see Lemma 2.3d}) . \end{cases}$$

Then $u - u_\infty$ and $\sqrt{t}\,(u - u_\infty) \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$; consequently $u(t) \to u_\infty$ strongly as $t \to \infty$ and $|u(t) - u_\infty| = O\left(\frac{1}{\sqrt{t}}\right)$ as $t \to \infty$.

Remark 2.7. Since $b \equiv B$ satisfies the hypothesis of Lemma 2.3, B is strongly positive on $[0,\infty)$, and therefore $\int_0^\infty B(t)dt > 0$. Since the operator A is maximum monotone on H the

limit equation (V_L) has a unique solution for any $f_\infty \in H$; in particular, if $f_\infty = 0$, $u_\infty = 0$ and in this case Theorem 2.6 reduces to Corollary 2.5.

Corollaries 2.4, 2.5 and Theorem 2.6 together form the natural generalization to Hilbert space of corresponding scalar results for (V) due to Levin [8] and Londen [9].

<u>Proof of Theorem 2.1.</u> (a) Take the scalar product of (2.1), equivalent to

$$\frac{1}{b(0)} \frac{du}{dt} + \frac{d}{dt} (k*u) + v = f_1 \qquad (v \in Au) , \tag{2.12}$$

with u and integrate from 0 to T. We obtain using (2.4), (2.5),

$$\frac{1}{2b(0)} |u(T)|^2 + ([illegible] + \varepsilon) \int_0^T |u(t)|^2 dt \le \frac{1}{2b(0)} |f(0)|^2 + \int_0^T (f_1(t),u(t))dt .$$

Since $[illegible] + \varepsilon > 0$, $f_1 \in L^2(0,\infty;H)$, one obtains by standard estimates $u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$.

(b) Next take the inner product of (2.12) with tu and integrate from 0 to T. Integrating $\int_0^T (tu(t), \frac{du}{dt}(t))dt$ by parts and using assumption (2.4) we obtain

$$\begin{aligned} &\frac{T}{2b(0)} |u(t)|^2 + \int_0^T t(u(t), \frac{d}{dt} (k*u)(t))dt + \varepsilon \int_0^T t|u(t)|^2 dt \\ &\qquad \le \frac{1}{2b(0)} \int_0^T |u(t)|^2 dt + \int_0^T (tf_1(t),u(t))dt . \end{aligned} \tag{2.13}$$

Now

$$\begin{aligned} \int_0^T t(u(t), \frac{d}{dt} (k*u)(t))dt &= \int_0^T k(0)t|u(t)|^2 dt + \int_0^T (t\, u(t),(k'*u)(t))dt \\ &= \int_0^T k(0)t|u(t)|^2 dt + \int_0^T (\sqrt{t}\, u(t),(k'*\sqrt{t}\, u)(t))dt \\ &\quad + \int_0^T (\sqrt{t}\, u(t), \int_0^t k'(t-\tau)(\sqrt{t}-\sqrt{\tau})u(\tau)d\tau)dt \\ &= \int_0^T (\sqrt{t}\, u(t), \frac{d}{dt} (k*\sqrt{t}\, u)(t))dt \\ &\quad + \int_0^T (\sqrt{t}\, u(t), \int_0^t k'(t-\tau)(\sqrt{t}-\sqrt{\tau})u(\tau)d\tau)dt = I + J . \end{aligned}$$

By assumption (2.5)

$$I = \int_0^T (\sqrt{t}\ u(t), \frac{d}{dt}(k*\sqrt{t}\ u)(t))dt \geq \eta \int_0^T t|u(t)|^2 dt .$$

We next estimate J as follows; noting that

$$\sqrt{t} - \sqrt{\tau} \leq \sqrt{t - \tau} \qquad (0 \leq \tau \leq t) ,$$

one has

$$|J| = |\int_0^T (\sqrt{t}\ u(t), \int_0^t k'(t-\tau)(\sqrt{t} - \sqrt{\tau})u(\tau)d\tau)dt$$

$$\leq \int_0^T t^{1/2}|u(t)| \left[\int_0^t \sqrt{t-\tau}\ |k'(t-\tau)||u(\tau)|d\tau\right] dt .$$

Since $u \in L^2(0,\infty; H)$ by part (a), $\sqrt{t}\ k' \in L^1(0,\infty)$ we obtain

$$|J| \leq \left(\int_0^T t|u(t)|^2 dt\right)^{1/2} \|u\|_{L^2(0,\infty;H)} \int_0^\infty \sqrt{t}|k'(t)|dt .$$

Thus

$$(2.14) \qquad \int_0^T t(u(t), \frac{d}{dt}(k*u)(t))dt \geq \eta \int_0^T t|u(t)|^2 dt$$

$$- \|u\|_{L^2(0,\infty;H)} \|\sqrt{t}\ k'\|_{L^1(0,\infty)} \left(\int_0^T t|u(t)|^2 dt\right)^{1/2} .$$

Using (2.14) in (2.13) we obtain the final a priori estimate

$$\frac{T}{2b(0)}|u(t)|^2 + (\eta + \varepsilon)\int_0^T t|u(t)|^2 dt \leq \frac{1}{2b(0)} \|u\|^2_{L^2(0,\infty;H)}$$

$$+ \|u\|_{L^2(0,\infty;H)} \|\sqrt{t}\ k'\|_{L^1(0,\infty)} \left(\int_0^T t|u(t)|^2 dt\right)^{1/2} + \|\sqrt{t}\ f_1\|_{L^2(0,\infty;H)} \|u\|_{L^2(0,\infty;H)} .$$

Since $\eta + \varepsilon > 0$, $u \in L^2(0,\infty;H)$ by part (a), and by assumption $\sqrt{t}\ f_1 \in L^2(0,\infty;H)$, $\sqrt{t}\ k' \in L^1(0,\infty)$, the conclusions $\sqrt{t}\ u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$ follow in a standard manner. This completes the proof.

Proof of Theorem 2.6. The proof will be reduced to that of Theorem 2.1 by the following steps. First by Lemma 2.3 $\int_0^\infty B(t)dt = k_\infty^{-1} > 0$. Therefore, the limit equation (V_L) can be written in the form

$$k_\infty u_\infty + Au_\infty \ni k_\infty f_\infty ,$$

which is the same as

$$\frac{1}{b(0)} \frac{d}{dt} u_\infty + \frac{d}{dt} (k*u_\infty) + Au_\infty \ni k_\infty f_\infty + (k(t) - k_\infty)u_\infty . \tag{2.15}$$

Next, subtracting (2.15) from (2.1) gives

$$\frac{1}{b(0)} \frac{d}{dt} (u - u_\infty) + \frac{d}{dt} k*(u - u_\infty) + Au - Au_\infty \ni F_1(t) \quad (0 < t < \infty) , \tag{2.16}$$

where by an elementary calculation

$$F_1(t) = \frac{1}{b(0)} F'(t) + k(0)F(t) + K(t)f_\infty + (k'*F)(t) - u_\infty K(t) . \tag{2.17}$$

Lemma 2.3 and the assumptions concerning F clearly imply that F_1 satisfies the same assumptions as f_1 in Theorem 2.1. The method of proof of Theorem 2.1 applied to (2.16), (2.17), where the coercivity assumption (2.11) is used in place of (2.4), now yields the needed *a priori* estimates for $u - u_\infty$ and $\sqrt{t}\,(u - u_\infty)$, and completes the proof.

Example 2.8. We give an example of a maximum monotone operator A in (V) which is not a subdifferential, and for which the theory developed in this section is applicable. Let $\Omega \subset R^n$ be a bounded open set with smooth boundary $\partial\Omega$. Let H be the Hilbert space $L^2(\Omega)$. Let β be a maximum monotone graph with $0 \in \beta(0)$ and with primitive j (i.e. $\beta = \partial j$). Let A_1 be the operator defined by

$$D(A_1) = \{u : u \in H_1^0(\Omega) \cap H^2(\Omega),\ \beta(u) \in L^2(\Omega)\} ,$$

$$A_1 u = -\Delta u + \beta(u) \qquad (u \in D(A_1)) .$$

It is clear that A_1 is maximum monotone on H since $A_1 = \partial\varphi_1$, where $\varphi_1 : H \to (-\infty,\infty]$ is the proper, convex, l.s.c. function given by

$$\varphi_1(u) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega j(u)dx & \text{if } u \in H_0^1(\Omega) \text{ and } j(u) \in L^1(\Omega) \\ +\infty & \text{otherwise} . \end{cases}$$

Define $L(u) = \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i}$ ($b_i \in R$, $u \in H_0^1(\Omega)$). By the divergence theorem $L(u)$ is monotone and $(u,L(u)) = \int_\Omega uL(u)dx = 0$. Finally, following Pazy [17, Ex. 3.5] define

$$A = A_1 + L .$$

By a perturbation theorem of Crandall and Pazy [6], A is maximum monotone on $H = L^2(\Omega)$, and by an easy calculation using Green's theorem and the Poincaré inequality there exists a constant $\varepsilon > 0$ such that

$$(Au,u) = (A_1u,u) = -\int_\Omega u\Delta u dx + \int_\Omega u\beta(u)dx \geq \int_\Omega |\nabla u|^2 dx \geq \varepsilon \|u\|^2_{L^2(\Omega)} .$$

Thus A satisfies the coercivity assumption (2.4) for every $T > 0$.

3. Boundedness and Asymptotic Properties When $A = \partial\varphi$. Let the general assumptions (H_b), (H_φ), (H_f) be satisfied and let u be a strong or generalized solution of (V) on $[0,\infty)$. In this section we shall obtain different boundedness and asymptotic results for the case $A = \partial\varphi$, and when $b_\infty > 0$ in (H_b). These results, motivated by the physical problem discussed in Section 4, are deduced from a priori estimates which are obtained directly from the equivalent Cauchy problem.

$$\text{(V')} \qquad \frac{du}{dt} + b(0)Au + B' * Au \ni F' \quad (0 < t < \infty), \quad u(0) = f(0) .$$

Theorem 3.1. *Let the general assumptions* (H_b) *with* $b_\infty > 0$, (H_φ), (H_f) *be satisfied and let* u *be a strong or generalized solution of* (V) *on* $[0,\infty)$. *If the kernel* b *satisfies the frequency domain condition* (F) *of Lemma 2.2 and if*

$$\text{(3.1)} \qquad \inf_{z\in H} \varphi(z) > -\infty ,$$

then

$$\text{(3.2)} \qquad \sup_{0\le t<\infty} \varphi(u(t)) < \infty ;$$

if $v \in \partial\varphi(u)$, *then*

$$\text{(3.3)} \qquad v \in L^2(0,\infty;H) ,$$

$$\text{(3.4)} \qquad \frac{du}{dt} \in L^2(0,\infty;H) ,$$

and

(3.5) u *is strongly uniformly continuous on* $[0,\infty)$.

If also $\lim_{|u|\to\infty} \varphi(u) = +\infty$, *then*

$$\text{(3.6)} \qquad \sup_{0\le t<\infty} |u(t)| < \infty ,$$

and

$$\text{(3.7)} \qquad \lim_{t\to\infty} \varphi(u(t)) = \varphi_\infty = \inf_{z\in H} \varphi(z) \quad \textit{exists} .$$

Moreover, if the inclusion $\partial\varphi(w) \ni 0$ *implies* $w = 0$, *then*

$$\text{(3.8)} \qquad u(t) \longrightarrow 0 \quad \text{(weakly) as} \quad t \to \infty .$$

The frequency domain condition (F) is satisfied by several classes of kernels b with $b_\infty > 0$ as was seen in Section 2 (see examples of $b = b_\infty + B$ with B given by (2.7), (2.9), (2.10)). Thus Theorem 3.1 generalizes a recent result of S. O. Londen [10, Corollary 2] and a result of V. Barbu [1, Theorem 2].

The assumptions concerning φ in Theorem 3.1 are not sufficient to obtain strong convergence of $u(t)$ to zero as $t \to \infty$. For this result one needs the coercivity condition (2.4) with $\varepsilon > 0$. If (2.4) is satisfied with $v \in \partial\varphi(u)$ it is a standard result (see Brézis [3]) that the inclusion $\partial\varphi(w) \ni 0$ has $w = 0$ as the only solution, and that $0 \in D(\partial\varphi)$. Then the definition of the subdifferential [3] implies that

$$\varphi(u) \geq \varphi(0) \qquad (u \in H) ,$$

and therefore assumption (3.1) of Theorem 3.1 holds. This motivates the following results which complement Corollary 2.4 for the case $A = \partial\varphi$. Note that in Theorem 3.2 below only the frequency domain condition (F), but not the assumption that B is a kernel of positive type (see Lemma 2.2), is needed. Also note that here the assumption on F is less restrictive.

Theorem 3.2. Let the general assumptions (H_b) with $b_\infty > 0$, (H_φ), (H_f) be satisfied, and let u be a strong or generalized solution of (V) on $[0,\infty)$. Let b satisfy the frequency domain condition (F), and for $v \in \partial\varphi(u)$ let the coercivity condition (2.4) with $\varepsilon > 0$ be satisfied. Then conclusions (3.2)-(3.5) of Theorem 3.1 hold, and $u \in L^2(0,\infty;H)$, which implies that $u(t) \to 0$ strongly as $t \to \infty$.

Remark 3.3. If $b(t) \equiv b_\infty > 0$ in (V), a case not excluded in Theorems 3.1 and 3.2, the above theorem and its proof yield a simple boundedness and asymptotic behaviour result for the evolution equation

$$\frac{du}{dt} + b_\infty \partial\varphi(u) \ni g, \quad u(0) = u_0 ,$$

where $g = F'$; compare Brézis [3, Theorem 3.11] where $g \in L^1(0,\infty;H)$.

Remark 3.4. If the coercivity condition (2.4) with $\varepsilon > 0$ and $A = \partial\varphi$ is replaced by the more general condition: for every $T > 0$ there exists $\varepsilon > 0$ such that

$$\text{(2.4')} \qquad \text{if } v \in \varphi(u), \text{ then } \int_0^T (v(t),u(t) - z)dt \geq \int_0^T |u(t) - z|^2 dt$$

for some $z \in H$, then it is easy to show that the inclusion $\partial\varphi(w) \ni 0$ has $w = z$ as the only solution, and that

$$\varphi(u) \geq \varphi(z) \qquad (u \in H) .$$

Then the method of proof of Theorem 3.2 easily yields that $u(t) \to z$ strongly as $t \to \infty$.

Remark 3.5. In Theorem 3.1 and 3.2 the assumption $b_\infty > 0$ in (H_b) is crucial; if $b_\infty = 0$ the frequency domain condition (F) cannot be satisfied (see examples (2.7), (2.9), (2.10)) for any $\delta > 0$. On the other hand, in these theorems f_∞ in (H_f) is arbitrary and the case $f_\infty = 0$ is not ruled out, provided $b_\infty > 0$. If $b_\infty = 0$ in (H_b), one can, of course, still apply Corollary 2.5 if $f_\infty = 0$, and Theorem 2.6 if $f_\infty \neq 0$, with $A = \partial\varphi$.

Proof of Theorem 3.1. (a) Let $0 < T < \infty$ be arbitrary; take the scalar product of (V') with $v \in \partial\varphi(u)$ and integrate over $[0,T]$. Using $\frac{d}{dt}\varphi(u(t)) = (v(t), \frac{du}{dt}(t))$ a.e. (see Brézis [3]) one obtains

$$(3.9) \qquad \varphi(u(T)) + b(0)\int_0^T |v(t)|^2 dt + Q_{B'}[v;T] = \int_0^T (F'(t),v(t))dt + \varphi(f(0)) ,$$

where

$$Q_{B'}[v;T] = \int_0^T (v(t),B'*v(t))dt .$$

We next apply a frequency domain method (see Nohel and Shea [14]) to $Q_{B'}$. Define $v_T(t) = v(t)\chi[0,T]$ and its Fourier transform

$$\tilde{v}_T(\eta) = \int_{-\infty}^{\infty} e^{-i\eta t} v_T(t)dt .$$

Extend B' evenly to $(-\infty,0)$ by $B'(-t) = B'(t)$ $(0 \leq t < \infty)$. In the following calculation use is made of the hypothesis $B' \in L^1(0,\infty)$, the Parseval and convolution theorems:

$$Q_{B'}[v;T] = \int_0^T (v(t), B'*v(t))dt = \frac{1}{2}\int_0^T (v(t), \int_0^T B'(t-\tau)v(\tau)d\tau)dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} (v_T(t), \int_{-\infty}^{\infty} B'(t-\tau)v_T(\tau)d\tau)dt = \frac{1}{4\pi}\int_{-\infty}^{\infty} |\tilde{v}_T(\eta)|^2 \tilde{B}'(\eta)d\eta \ .$$

Since B' is even, $\tilde{B}'(\eta) = 2\text{Re}\,\hat{B}'(i\eta)$ $(\eta \in R)$, where $\hat{\cdot}$ denotes the Laplace transform. The assumptions $B, B' \in L^1(0,\infty)$ and the familiar formula $\hat{B}'(i\eta) = i\eta\,\hat{B}(i\eta) - B(0)$ yield $\text{Re}\,\hat{B}'(i\eta) = -\eta\text{Im}\,\hat{B}(i\eta) - B(0)$. Therefore,

$$Q_{B'}[v;T] = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{v}_T(\eta)|^2[-\eta\text{Im}\,\hat{B}(i\eta) - B(0)]d\eta \ .$$

Substituting this result into (3.9) and using $\int_0^T |v(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{v}_T(\eta)|^2d\eta$, as well as $b(0) - B(0) = b_\infty$, the frequency domain condition (F), and Parseval's theorem again, yields

$$(3.10) \quad \varphi(u(T)) + \delta\int_0^T |v(t)|^2dt \le |\varphi(f(0))| + \int_0^T |(F'(t),v(t))|dt \quad (0 \le T < \infty) \ .$$

The assumption $F' \in L^2(0,\infty)$, Cauchy-Schwartz and an elementary inequality give the estimate

$$(3.11) \quad \varphi(u(T)) + \frac{\delta}{2}\int_0^T |v(t)|^2dt \le |\varphi(f(0))| + \frac{1}{2\delta}\int_0^\infty |F'(t)|^2dt < \infty \quad (0 < T < \infty) \ .$$

Assumption (3.1) used in (3.11) yields conclusions (3.2), (3.3) and (3.6).

Returning to (V') and using $B' \in L^1(0,\infty)$, $v \in L^2(0,\infty;H)$, $F' \in L^2(0,\infty;H)$ gives conclusion (3.4). Combining (3.3), (3.4) with $\frac{d}{dt}\varphi(u(t)) = (v(t), \frac{du}{dt}(t))$ yields $\frac{d}{dt}\varphi(u(t)) \in L^1(0,\infty)$, and this together with assumption (3.1) implies that $\lim_{t\to\infty}\varphi(u(t))$ exists. To establish all of (3.7) we use the definition of subdifferential: for every $v \in \partial\varphi(u)$ and for every $w \in H$ $\varphi(u(t)) \le \varphi(w) + (v(t),u(t) - w)$, $0 \le t < \infty$. Since

$u \in L^\infty(0,\infty;H)$ and $v \in L^2(0,\infty;H)$ there exists a sequence $\{t_n\} \to \infty$ as $n \to \infty$ such that $(v(t_n), u(t_n) - w) \to 0$ as $n \to \infty$; this proves (3.7), and from it easily (3.8). To prove (3.5) take $\tau < t$ and use (3.4) and Cauchy-Schwartz obtaining:

$$|u(t) - u(\tau)| \leq \int_\tau^t \left|\frac{du}{dt}(s)\right| ds \leq \sqrt{t-\tau}\left\{\int_0^\infty \left|\frac{du}{dt}(s)\right|^2 ds\right\}^{1/2}$$

$$\leq K\sqrt{t-\tau} \qquad (0 \leq \tau < t < \infty) .$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. As remarked in the paragraph preceding Theorem 3.2 the coercivity condition (2.4) implies that

$$\operatorname*{Inf}_{z \in H} \varphi(z) \geq \varphi(0) > -\infty ,$$

so that assumption (3.1) is satisfied. Thus conclusions (3.2)-(3.5) follow immediately from Theorem 3.1. In view of (3.5) the conclusion $u(t) \to 0$ strongly as $t \to \infty$ follows once it is shown that $u \in L^2(0,\infty;H)$. But using assumption (2.4) with $\varepsilon > 0$ and $v \in L^2(0,\infty;H)$ for $v \in \partial\varphi(u)$ (proved in (3.3)) one has

$$\varepsilon \int_0^T |u(t)|^2 dt \leq \int_0^T (v(t),u(t))dt \leq \frac{\varepsilon}{2}\int_0^T |u(t)|^2 dt + \frac{1}{2\varepsilon}\int_0^T |v(t)|^2 dt .$$

Thus

$$\frac{\varepsilon}{2}\int_0^T |u(t)|^2 dt \leq \frac{1}{2\varepsilon}\int_0^\infty |v(t)|^2 dt < \infty \qquad (0 < T < \infty) .$$

Since $\varepsilon > 0$, this completes the proof of Theorem 3.2.

We close this section by giving two typical examples of operators $A = \partial\varphi$ arising in heat flow, and satisfying the assumptions of the theory developed above. The first is in one space dimension, the second in several space dimensions.

Example 3.6. Let $H = L^2(0,1)$ be the real Hilbert space of square integrable functions on $(0,1)$. Let $\sigma : \mathbf{R} \to \mathbf{R}$ satisfy the assumptions

$$(\sigma) \qquad \sigma \in C^1(\mathbf{R}),\ \sigma(0) = 0,\ \sigma'(\xi) \geq p_0 > 0 \quad (\xi \in \mathbf{R}),$$

for some $p_0 > 0$. Define $W : \mathbf{R} \to \mathbf{R}^+$ by

$$W(r) = \int_0^r \sigma(\xi)d\xi \quad \left(\geq \frac{p_0}{2} r^2\right) \qquad (r \in \mathbf{R}),$$

and define $\varphi : H \to (-\infty,+\infty]$ by

$$\varphi(y) = \begin{cases} \int_0^1 W\left(\frac{dy}{dx}(x)\right)dx & \text{if } y \in H_0^1(0,1) \\ +\infty & \text{otherwise}. \end{cases}$$

Then it is readily verified that φ is convex, l.s.c. and proper on H, and $Ay = \partial\varphi(y) = -\frac{d}{dx}\sigma\left(\frac{dy}{dx}\right)$ with

$$D(\partial\varphi) = \left\{y \in H_0^1(0,1) : \frac{d}{dx}\sigma\left(\frac{dy}{dx}\right) \in L^2(0,1)\right\}.$$

Let $y \in H_0^1(0,1)$; then from the definition of φ and the Poincaré inequality one has

$$\varphi(y) \geq \frac{p_0}{2}\int_0^1 \left|\frac{dy}{dx}(x)\right|^2 dx \geq \frac{p_0}{2}\pi^2 \int_0^1 |y(x)|^2 dx \geq 0.$$

Thus φ satisifes assumptions (3.1) and $\lim_{|y|\to\infty} \varphi(y) = \infty$ of Theorem 3.1. Moreover

$$(y,\partial\varphi(y)) = -\int_0^1 y(x)\frac{d}{dx}\sigma\left(\frac{dy}{dx}(x)\right)dx,$$

and an integration by parts, $y \in H_0^1(0,1)$ and the Poincaré inequality give

$$(y,\partial\varphi(y)) = \int_0^1 \frac{dy}{dx}(x)\sigma\left(\frac{dy}{dx}(x)\right)dx \geq p_0\int_0^1 \left|\frac{dy}{dx}(x)\right|^2 dx \geq p_0\pi^2\int_0^1 |y(x)|^2 dx.$$

Thus $Ay = \partial\varphi(y)$ satisfies the coercivity assumption (2.4) for every $0 < T < \infty$ with $\epsilon = p_0\pi^2 > 0$. Therefore, this operator A satisfies the assumptions of Theorems 3.1 and 3.2.

Example 3.7. Let Ω be a bounded domain in $\mathbf{R}^n$ (for heat flow $n = 2$ or 3) with smooth boundary Γ. Let $\lambda : \mathbf{R}^+ \to \mathbf{R}$ be a given smooth function satisfying the assumption

$$(\lambda) \qquad \begin{cases} \lambda(0) > 0, \text{ there exists } p_0 > 0 \text{ such that } \lambda(\xi) \geq p_0 \text{ and} \\ \xi\lambda'(\xi) + \lambda(\xi) \geq p_0 \quad (\xi \in \mathbf{R}) . \end{cases}$$

Define $\Lambda : \mathbf{R} \to \mathbf{R}^+$ by

$$\Lambda(r) = \int_0^r \xi\lambda(\xi)d\xi \qquad \left(\geq \frac{p_0}{2} r^2\right) \qquad (r \in \mathbf{R}) .$$

Let $H = L^2(\Omega)$ and define

$$\varphi(u) = \begin{cases} \int_\Omega \Lambda(|\nabla u|)dx & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{otherwise} . \end{cases}$$

Then it is readily verified that $\varphi : H \to (-\infty,+\infty]$ is convex, l.s.c., and proper on H and

$$Au = \partial\varphi(u) = -\nabla\cdot(\lambda(|\nabla u|)\nabla u)$$

with

$$D(\partial\varphi) = \{u \in H_0^1(\Omega) : \nabla\cdot(\lambda(|\nabla u|)\nabla u) \in L^2(\Omega)\} .$$

Clearly $\varphi(u) \geq 0$ $(u \in H)$ and by the Poincaré inequality $\varphi(u) \to \infty$ as $|u| \to \infty$; thus φ satisfies assumptions Theorem 3.1. Using integration by parts and the Poincaré inequality one also has

$$(Au,u) \geq kp_0|u|_H^2 ,$$

where $k > 0$ is the constant in the Poincaré inequality. Thus $Au = \partial\varphi(u)$ also satisfies the coercivity assumption (2.4) in Theorem 3.2 with $\varepsilon = kp_0 > 0$.

4. Nonlinear Heat Flow in a Material with Memory. Consider nonlinear heat flow in a homogeneous bar of unit length of material with memory with the temperature $u = u(t,x)$ maintained at zero at $x = 0$ and $x = 1$. We shall assume that the history of u is prescribed for $t \leq 0$ and $0 \leq x \leq 1$. The equation satisfied by u in such a material is derived from the assuptions that the interval energy ε and the heat flux q are functionals (rather than functions) of u and of the gradient of u respectively. According to the theory developed by Coleman, Gurtin, Knoll, Pipkin, Mac Camy and Nunziato (see e.g. Mac Camy [11], [12] and Nunziato [15]) for heat flow in materials of fading memory type the functionals ε and q are taken respectively as:

$$(4.1) \qquad \varepsilon(t,x) = b_0 u(t,x) + \int_0^t \beta(t-s)u(s,x)ds \quad (t \geq 0,\ 0 < x < 1)\ ,$$

$$(4.2) \qquad q(t,x) = -c_0\sigma(u_x(t,x)) + \int_0^t \gamma(t-s)\sigma(u_x(s,x))ds \quad (t \geq 0,\ 0 < x < 1)\ .$$

In writing the functionals ε and q we have assumed without loss of generality that the history of the temperature u is prescribed as zero for $t < 0$ (if this is not the case and if the history of u is sufficiently smooth for $t < 0$ and $0 < x < 1$, this has the effect of altering the forcing term h in equation (4.3) below - and consequently also G in (4.5) below - by additional known forcing terms). In (4.1), (4.2) $b_0 > 0$, $c_0 > 0$ are given constants, $\beta, \gamma : [0,\infty) \to R$ are given sufficiently smooth functions (called the internal energy and heat flux relaxation functions respectively). In the physical literature β, γ are usually assumed to be decaying exponentials with positive coefficients. As we shall see the theory developed here permits a much greater generality, and we shall require for physical reasons that $\beta(0) > 0$, $\gamma(0) > 0$, that β and γ $L^1(0,\infty)$, that

$$b_0 + \int_0^t \beta(\tau)d\tau > 0, \quad c_0 - \int_0^t \gamma(\tau)d\tau > 0 \quad (0 \leq t < \infty)\ ,$$

as well as that the conditions

(PW) $$b_0 + \mathrm{Re}\,\hat{\beta}(i\eta) > 0 \qquad (\eta \in R),$$

(•) $$c_0 - \int_0^\infty \gamma(\tau)d\tau > 0,$$

where $\hat{\beta}(i\eta) = \int_0^\infty \beta(t)\exp(-i\eta t)dt$, are satisfied. The above assumptions will be motivated presently. Remark 4.10 below shows that the physically reasonable assumptions $b_0 + \int_0^t \beta(\tau)d\tau > 0$ and $c_0 - \int_0^t \gamma(\tau)d\tau > 0$ $(0 \leq t < \infty)$ are not essential for the theory developed here to apply. The real function $\sigma : R \to R$ in (4.2) will be assumed to satisfy the assumptions (σ) of Example 3.6. It should be noted that the case $\sigma(r) \equiv r$ gives rise to the linear model derived in Nunziato [15], and that (4.2) is one reasonable generalization of the heat flux for nonlinear heat flow in one space dimension.

If $h = h(t,x) \in L^1_{loc}(0,\infty;L^2(0,1))$ represents the external heat supply added to the rod for $t \geq 0$ and $0 < x < 1$, and if $u(0,x) = u_0(x)$, $0 < x < 1$, *is the given initial* temperature distribution, the law of balance of heat $(\varepsilon_t = -\mathrm{div}\,\underline{q} + h)$ shows that in one space dimension the temperature u satisfies the initial-boundary value problem

(4.3) $$\begin{cases} \dfrac{\partial}{\partial t}\,[b_0 u + \beta * u] = c_0\sigma(u_x)_x - \gamma * \sigma(u_x)_x + h & (0 < t < \infty,\ 0 < x < 1) \\ u(0,x) = u_0(x) \quad (0 < x < 1), & u(t,0) = u(t,1) \equiv 0 \quad (t \geq 0), \end{cases}$$

where subscripts denote differentiation with respect to x. Note that in an ordinary material $\beta = \gamma \equiv 0$, and (4.3) becomes the nonlinear diffusion equation in one space dimension.

The next task is to transform (4.3) to the equivalent form (V) which will be used for the analysis. Define

(4.4) $$C(t) = c_0 - \int_0^t \gamma(\tau)d\tau \qquad (0 \leq t < \infty),$$

(4.5) $$G(t,x) = b_0 u_0(x) + \int_0^t h(\tau,x)d\tau \qquad (0 \leq t < \infty,\ 0 < x < 1).$$

Noting that

$$\frac{\partial}{\partial t}(C*\sigma(u_x)_x)(t,x) = c_0\sigma(u_x(t,x))_x - (\gamma*\sigma(u_x)_x)(t,x) ,$$

and integrating (4.3) using the initial condition, and (4.5) yields the equivalent Volterra equation (to (4.3)):

(4.6) $b_0 u(t,x) + (\beta*u)(t,x) = (C*\sigma(u_x)_x)(t,x) + G(t,x) \quad (0 < t < \infty,\ 0 < x < 1)$,

where u satisfies the boundary conditions $u(t,0) = u(t,1) \equiv 0$ $(t \geq 0)$. Letting $A = \partial\varphi$ be the nonlinear operator defined in Example 3.6 under assumptions (σ) , the equation (4.6) has the abstract form

(V_1) $b_0 u + \beta*u + C*Au = G \quad (0 \leq t < \infty)$,

where $\varphi : H \to (-\infty,\infty]$ is the specific proper, convex, l.s.c. function defined in Example 3.6 on $H = L^2(0,1)$. To transform (V_1) to the equivalent form (V) define $\rho : [0,\infty) \to R$ to be the unique solution of the linear Volterra equation (called resolvent kernel of β):

(ρ) $b_0\rho(t) + (\beta*\rho)(t) = -\dfrac{\beta(t)}{b_0} \quad (0 \leq t < \infty)$.

It is standard that if $b_0 > 0$ and $\beta \in L^1_{loc}[0,\infty)$, equation ($\rho$) has a unique solution $\rho \in L^1_{loc}(0,\infty)$. Applying the variation of constants formula for Volterra equations [13]

$$(b_0 y + \beta*y = g \iff y = \frac{g}{b_0} + \rho*g)$$

finally yields that (V_1) is equivalent to (V) with the definitions

(b) $b(t) = \dfrac{C(t)}{b_0} + (\rho*C)(t) \quad (0 \leq t < \infty)$

(f) $f(t) = \dfrac{G(t,\cdot)}{b_0} + (\rho*G)(t,\cdot) \quad (0 \leq t < \infty)$.

Similar considerations show that for heat flow in a bounded homogeneous body Ω of isotropic material with memory in $\mathbf{R}^2$ or $\mathbf{R}^3$ with a smooth boundary Γ, the temperature u will also satisfy the abstract Volterra equation (V) with the kernel b and forcing term f given exactly as above, but with the nonlinear operator $A = \partial\varphi$ defined as in Example 3.7 with $H = L^2(\Omega)$.

We next comment on the significance of the assumptions concerning β, γ as well as (PW) and (γ). Since the relaxation functions β and γ are generally taken as decaying exponentials with positive coefficients in the physical literature, it is certainly reasonable to assume that $\beta, \gamma \in L^1(0,\infty)$ and that $\beta(0) > 0$, $\gamma(0) > 0$. We next motivate the assumption that $b_0 + \int_0^t \beta(\tau)d\tau > 0$ $(0 \leq t < \infty)$. A similar reasoning motivates $c_0 - \int_0^t \gamma(\tau)d\tau > 0$ $(0 \leq t < \infty)$. Consider the internal energy ε defined by (4.1) and suppose that the temperature u is maintained at zero up to time t_0 and at a state $\bar{u}_1(x) > 0$ $(0 < x < 1)$ for $t > t_0$. One would then expect the internal energy to be positive for $t > t_0$. If the function β is positive for $t \geq 0$ this is automatically the case. However if not, the assumption $b_0 + \int_0^t \beta(\tau)d\tau > 0$ $(0 \leq t < \infty)$ is natural in view of the fact that in this situation

$$\varepsilon(t,x) = \overline{u_1}(x)(b_0 + \int_{t_0}^t \beta(\tau)d\tau) \qquad (t_0 \leq t < \infty) .$$

Since $\beta \in L^1(0,\infty)$ equation (4.1) shows that ε is bounded whenever u is bounded. The assumption (PW) implies that $b_0 + \int_0^\infty \beta(t)dt > 0$ (take $\eta = 0$); thus if $u(x,t)$ tends to an equilibrium state $\bar{u}(x) > 0$ as $t \to \infty$, (4.1) implies that the corresponding limiting internal energy $\bar{\varepsilon}(x) > 0$ as is to be expected. For physical reasons it is also to be expected that if ε is bounded the temperature should be bounded. Applying the variations of constants formula to (4.1) yields

$$u(t,x) = \frac{\varepsilon(t,x)}{b_0} + (\rho * \varepsilon)(t,x) \qquad (0 \leq t < \infty,\ 0 < x < 1) ,$$

where ρ is the resolvent kernel of β defined by equation (ρ). Thus to have u bounded whenever ε is bounded it is sufficient to require that $\rho \in L^1(0,\infty)$. But by the Paley-Wiener theorem [16] applied to equation (ρ), $\beta \in L^1(0,\infty)$ implies that $\rho \in L^1(0,\infty)$ if and only if

$$b_0 + \hat{\beta}(z) \neq 0 \quad \text{for} \quad \operatorname{Re} z \geq 0 .$$

The condition (PW) now results from taking the real part of this expression, noting that

for physical reasons one wants $b_0 + \hat{\beta}(0) > 0$, and arguing as in the proof of Lemma 2.2.

To motivate assumption (γ) suppose that $u(t,x) \to \bar{u}(x)$ as $t \to \infty$ and that $\frac{d}{dx}\,\bar{u}(x) > 0$, implying that $\sigma\left(\frac{d\bar{u}}{dx}\right) > 0$. One then expects that the limiting heat flux $\bar{q}(x)$ in equation (4.2) is strictly negative, if the process being modelled represents "forward" heat flow; condition (γ) insures that this is the case.

In order to apply the theory of Section 3 to the Volterra equation (V) with the kernel b and forcing function f defined by equations (b) and (f) respectively, we must impose some additional technical assumptions on the functions β, γ, $u_0(x)$, and h in order that the assumptions of Theorems 3.1 and 3.2 are satisfied. Our main result is Theorem 4.7 below.

First, concerning the kernel b defined by (b) we have the following simple result which insures that b satisfies assumption (H_b); its elementary proof is omitted.

Lemma 4.1. *Let* $b_0 > 0$, $c_0 > 0$, β, γ, $t\beta$, $t\gamma \in L^1(0,\infty)$, *and let assumptions* (PW) *and* (γ) *be satisfied. Define*

$$(4.7)\qquad b_\infty = \frac{c_0 - \int_0^\infty \gamma(t)\,dt}{b_0 + \int_0^\infty \beta(t)\,dt},$$

$$(4.8)\qquad B(t) = \frac{C(t)}{b_0} + (\rho * C)(t) - b_\infty,$$

where $C(t) = c_0 - \int_0^t \gamma(\tau)\,d\tau$, *and* ρ *is the resolvent of* β *uniquely defined by equation* (ρ). *Then* $b_\infty > 0$ *and* $B, B' \in L^1(0,\infty)$, *and* $b(t) = b_\infty + B(t)$ *satisfies* (H_b) *with* $b(0) = \frac{c_0}{b_0} > 0$, $B(0) = \frac{c_0}{b_0} - b_\infty > 0$.

The next elementary result gives sufficient conditions in order that the forcing function f in (V) defined by equation (f) satisfies assumption (H_f).

Lemma 4.2. *Let* $H = L^2(0,1)$ *and* $u_0 \in H_0^1(0,1)$. *Let* $\beta \in L^1(0,\infty) \cap L^2(0,\infty)$ *and let assumption* (PW) *be satisfied. Finally, assume that*

(h) $$h \in L^1(0,\infty;H) \cap L^2(0,\infty;H).$$

Then the function $f : [0,\infty) \times (0,1) \to H$, defined by equations (f), [illegible], where ρ is the resolvent of β, satisfies $f \in W^{1,2}_{loc}(0,\infty;H)$ and $f(0,x) = u_0(x) \in H^1_0(0,1)$. Moreover,

$$f(t,x) = f_\infty(x) + F(t,x) \qquad (0 \le t < \infty,\ 0 < x < 1),$$

where

$$f_\infty(x) = \Big(b_0 u_0(x) + \int_0^\infty h(\tau,x)d\tau\Big)\Big(\frac{1}{b_0} + \int_0^\infty \rho(\tau)d\tau\Big), \tag{4.9}$$

$$\begin{aligned} F(t,x) &= \frac{G(t,x)}{b_0} + (\rho * G)(t,x) - f_\infty(x) = -\frac{1}{b_0}\int_t^\infty h(\tau,x)d\tau \\ &\quad - \int_0^t \rho(t-s)\int_s^\infty h(\tau,x)d\tau ds - \int_t^\infty \rho(\tau)d\tau\Big(b_0 u_0(x) + \int_0^\infty h(\tau,x)d\tau\Big), \end{aligned} \tag{4.10}$$

and $\frac{\partial F}{\partial t} \in L^2(0,\infty;H)$. If in addition $t\beta \in L^1(0,\infty)$ and $th \in L^1(0,\infty;H)$, then $F \in L^2(0,\infty;H)$.

Sketch of Proof of Lemma 4.2. The assumptions $\beta \in L^1(0,\infty)$ and (PW), together with the Paley-Wiener theorem [16], applied to the resolvent equation (ρ) imply that $\rho \in L^1(0,\infty)$. But then the assumption $\beta \in L^2(0,\infty)$ and the fact that $\rho \in L^1(0,\infty)$ imply that also $\rho \in L^2(0,\infty)$ from the resolvent equation. These facts combined with the definition of f in (f) and assumption (h) yield the formulae (4.9) for f_∞ and (4.10) for F given in the statement as well as $f \in W^{1,2}_{loc}(0,\infty;H)$. From formula (4.10) one easily proves that

$$\frac{\partial F}{\partial t}(t,x) = \frac{1}{b_0}h(t,x) + b_0 u_0(x)\rho(t) + (\rho * h)(t,x) \quad (0 \le t < \infty,\ 0 < x < 1); \tag{4.11}$$

then $\frac{\partial F}{\partial t} \in L^2(0,\infty;H)$ follows from $h \in L^2(0,\infty;H)$ and $\rho \in L^1(0,\infty) \cap L^2(0,\infty)$. Finally, (PW) and $t\beta \in L^1(0,\infty)$, together with $\rho \in L^1(0,\infty)$ imply that $t\rho \in L^1(0,\infty)$ from the resolvent equation. This, together with the assumption $th \in L^1(0,\infty)$ and routine estimates applied to the formula (4.10) yield $F \in L^2(0,\infty;H)$. This completes the proof.

In order to apply Theorems 3.1 and 3.2 to the physical problem of heat flow it remains to verify tht the assumptions (H_ω) and the coercivity assumptions are satisfied. As we have seen in Example 3.6 in the case of one space dimension (or Example 3.7 in the case of

two or three space dimensions), assumptions (σ) (or (λ) in the multidimensional case) imply that the function ϕ of Example 3.6 (or Example 3.7) convex, l.s.c., proper and satisfies $\phi(y) \geq 0$, $\phi(y) \to \infty$ as $|y| \to \infty$, $\inf_{y \in H} \phi(y) = 0$, and the inclusion $\partial\phi(w) \ni 0$ has $w = 0$ as the only solution; thus the assumptions concerning ϕ in Theorem 3.1 are satisfied. Moreover $A = \partial\phi$ satisfies the coercivity assumption (2.4) of Theorem 3.2 for every $T > 0$.

It remains to show that the frequency domain assumption (F) of Lemma 2.2 can be satisfied for physically reasonable classes of relaxation functions β, γ. In this direction we have:

Lemma 4.3. Let b_0, c_0, β, γ satisfy the assumptions of Lemma 4.1. Define the kernel b in (V) by equation (b). Then the frequency domain assumption (F) of Lemma 2.2 is equivalent to the condition: there exists $\delta > 0$ such that

$$\inf_{(\eta \in \mathbf{R})} \frac{(c_0 - \mathrm{Re}\hat{\gamma}(i\eta))(b_0 + \mathrm{Re}\hat{\beta}(i\eta)) - \mathrm{Im}\hat{\gamma}(i\eta)\mathrm{Im}\hat{\beta}(i\eta)}{|b_0 + \hat{\beta}(i\eta)|^2} \geq \delta . \tag{4.12}$$

Proof of Lemma 4.3. Define the constant $b_\infty > 0$ by (4.7) and the function B by (4.8) (see Lemma 4.1). Taking the Laplace transform of B one computes

$$\hat{B}(i\eta) = -\frac{i}{\eta}\left(\frac{c_0 - \hat{\gamma}(i\eta)}{b_0 + \hat{\beta}(i\eta)} - b_\infty\right) \qquad (\eta \in \mathbf{R}) .$$

Thus

$$b_\infty - \eta \,\mathrm{Im}\, \hat{B}(i\eta) = \mathrm{Re}\left(\frac{c_0 - \hat{\gamma}(i\eta)}{b_0 + \hat{\beta}(i\eta)}\right) \qquad (\eta \in \mathbf{R}) ,$$

from which the condition (4.12) is an immediate consequence.

Using Lemma 4.3 one can construct a large number of examples of functions β and γ such that assumption (F) is satisfied. In particular one has the following physically important special cases. Note that in Corollaries 4.4 and 4.6 below the physical

conditions $b_0 + \int_0^t \beta(\tau)d\tau > 0 \quad (0 \le t < \infty)$, $c_0 - \int_0^t \gamma(\tau)d\tau > 0 \quad (0 \le t < \infty)$ are both satisfied (although they are not explicitly needed in the theory), because the functions β and γ are positive, and assumption (γ) is assumed to hold. For a different example in which (γ) is satisfied but the above physical conditions need not hold see Remark 4.10 below.

Corollary 4.4. Let $b_0 > 0$, $c_0 > 0$ and $\beta, \gamma, t\beta, t\gamma \in L^1(0,\infty)$. Also assume that β and γ are positive, nonincreasing and convex on $[0,\infty)$, and that the assumption (γ) is satisfied. Then assumption (F) is satisfied if either for a fixed $b_0 > 0$ the constant $c_0 > 0$ is chosen sufficiently large, or if for a fixed $c_0 > 0$ the constant $b_0 > 0$ is sufficiently large.

Remark 4.5. (i) If $\beta = \gamma \equiv 0$ (the standard heat flow problem) (F) is satisfied for any choice of $b_0 > 0$, $c_0 > 0$ with $\delta = \frac{b_0}{c_0}$.

(ii) If $\beta \equiv 0$ and γ satisfies the assumptions of Corollary 4.4, (F) is satisfied for any choice of $b_0 > 0$, $c_0 > 0$ with $\delta = \dfrac{c_0 - \int_0^\infty \gamma(t)dt}{b_0}$.

(iii) If $\gamma \equiv 0$ and β satisfies the assumptions of Corollary 4.4, (F) is satisfied for any choice of $b_0 > 0$, $c_0 > 0$.

Sketch of Proof of Corollary 4.4. The proof will make use of Lemma 4.3; we establish (4.12). Since $\beta,\gamma \in L^1(0,\infty)$ and are positive, nonincreasing and convex, $\operatorname{Re} \hat{\beta}(i\eta)$ and $\operatorname{Re} \hat{\gamma}(i\eta)$ are nonnegative. The function

$$\operatorname{Im}\hat{\gamma}(i\eta)\operatorname{Im}\hat{\beta}(i\eta) = \int_0^\infty \gamma(t)\sin \eta t dt \int_0^\infty \beta(t)\sin \eta t dt \qquad (\eta \in \mathbf{R})$$

is even, continuous, zero when $\eta = 0$, nonnegative, and has limit zero as $\eta \to \infty$ (Riemann-Lebesgue lemma). The denominator in (4.12) satisfies

$$0 < b_0^2 \le |b_0 + \hat{\beta}(i\eta)|^2 \le 2b_0^2 + 3\left(\int_0^\infty \beta(t)dt\right)^2 \qquad (\eta \in \mathbf{R}) .$$

Moreover,

$$b_0 + \mathrm{Re}\,\hat{\beta}(i\eta) \geq b_0 > 0 \qquad (\eta \in \mathbf{R}) ,$$

(so that (PW) is satisfied), and

$$c_0 - \mathrm{Re}\,\hat{\gamma}(i\eta) \geq c_0 - \int_0^\infty \gamma(t)dt > 0 \qquad (\eta \in \mathbf{R}) .$$

Therefore, the existence of $\delta > 0$ such that (4.12) holds is established for choices of b_0 and c_0 as asserted. This completes the proof.

Another physically important case for the heat flow problem is the following special case of Lemma 4.3 and Corollary 4.4.

<u>Corollary 4.6</u>. <u>Let</u>

$$(4.13) \qquad \begin{cases} \beta(t) = \sum_{k=1}^{n} b_k e^{-\beta_k t} & (0 \leq t < \infty) , \\ \gamma(t) = \sum_{k=1}^{m} c_k e^{-\gamma_k t} & (0 \leq t < \infty) \end{cases}$$

<u>with</u> $b_k \geq 0$, $\beta_k \geq 0$, $c_k \geq 0$, $\gamma_k \geq 0$ <u>and strict inequalities hold for at least one pair</u> b_k, β_k <u>and one pair</u> c_k, γ_k. <u>Let</u> $b_0 > 0$, $c_0 > 0$, <u>and</u> $c_0 - \sum_{k=1}^{m} \frac{c_k}{\gamma_k} > 0$. <u>Then the frequency domain condition</u> (F) <u>is satisfied if</u>

$$(4.14) \qquad b_0\Big(c_0 - \sum_{k=1}^{m} \frac{c_k}{\gamma_k}\Big) > \frac{5}{4}\Big(\sum_{k=1}^{n} \frac{b_k}{\beta_k}\Big)\Big(\sum_{k=1}^{m} \frac{c_k}{\gamma_k}\Big) .$$

The proof of Corollary 4.6 is a consequence of showing that there exists a $\delta > 0$ such that (4.12) holds. The inequality (4.14) follows by using elementary calculus to find the infimum over $\eta \in \mathbf{R}$ of the expression in (4.12):

$$\frac{\Big(c_0 - \sum_{k=1}^{m} \frac{c_k\gamma_k}{\gamma_k^2 + \eta^2}\Big)\Big(b_0 + \sum_{k=1}^{n} \frac{b_k\beta_k}{\beta_k^2 + \eta^2}\Big) - \eta^2\Big(\sum_{k=1}^{m} \frac{c_k}{\gamma_k^2 + \eta^2}\Big)\Big(\sum_{k=1}^{n} \frac{b_k}{\beta_k^2 + \eta^2}\Big)}{\Big(b_0 + \sum_{k=1}^{n} \frac{b_k}{\beta_k^2 + \eta^2}\Big)^2 + \Big(\sum_{k=1}^{n} \frac{b_k\eta}{\beta_k^2 + \eta^2}\Big)^2} .$$

No claim is made that the constant $\frac{5}{4}$ in (4.14) is optimal.

We now combine Lemmas 4.1-4.3, and Corollaries 4.4 and 4.5 with the abstract theory to establish the following result for the physical heat flow problem (4.3) in a one-dimensional material with memory. To see that a more general result (not necessarily physical) with β and γ oscillatory can hold we refer to Remark 4.10 below.

Theorem 4.7. Let $b_0 > 0$, $c_0 > 0$, let $\beta, \gamma, t\beta, t\gamma \in L^1(0,\infty)$ and let $\dot\beta \in L^2(0,\infty)$. Assume that β and γ are positive, nondecreasing, convex, and that

$$(\gamma)\quad c_0 - \int_0^\infty \gamma(t)dt > 0 .$$

Assume that $\sigma : R \to R$ satisfies assumptions (σ) of Example 3.6, that the initial temperature $u_0 \in H_0^1(0,1)$, and that the external heat supply $h \in L^1(0,\infty;H) \cap L^2(0,\infty;H)$, where $H = L^2(0,1)$. Then the heat flow problem (4.3) has a unique strong solution u on $[0,\infty) \times (0,1)$ such that $\frac{\partial u}{\partial t} \in L^2_{loc}(0,\infty;H)$. Moreover, if either for a fixed $b_0 > 0$, the constant $c_0 > 0$ is sufficiently large, or for a fixed $c_0 > 0$, the constant $b_0 > 0$ is sufficiently large, then the solution u has the properties:

$$u \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H),\ \frac{du}{dt} \in L^2(0,\infty;H)\ ,$$

and $\lim_{t\to\infty} u(t) = 0$ strongly in H.

Remark 4.8. For heat flow in more than one space dimension let Ω be a bounded body in R^2 or R^3 with smooth boundary Γ. Then the temperature u satisfies an equation of the form (4.3) with the operator $-\sigma(u_x)_x$ replaced by $-\nabla\cdot(\lambda|\nabla u|)\nabla u)$; the boundary condition is $u(t,x) = 0$ $(0 \leq t < \infty,\ x \in \Gamma)$, and the initial condition is $u(0,x) = u_0(x)$ $(x \in \Omega)$. If H is the Hilbert space $L^2(\Omega)$, if the function $\lambda : R \to R^+$ satisfies assumptions (λ) of Example 3.7, if $u_0(x) \in H_0^1(\Omega)$, and $h \in L^1(0,\infty;H) \cap L^2(0,\infty;H)$, then the results of Theorem 4.7 holds, provided the constants $b_0 > 0$, $c_0 > 0$ and the relaxation functions β and γ satisfy the assumptions stated in Theorem 4.7.

Remark 4.9. Let

$$\beta(t) = \sum_{k=1}^{n} b_k e^{-\beta_k t} \qquad (0 \le t < \infty)$$

$$\gamma(t) = \sum_{k=1}^{m} c_k e^{-\gamma_k t} \qquad (0 \le t < \infty) ,$$

where $b_k \ge 0$, $\beta_k \ge 0$, $c_k \ge 0$, $\gamma_k \ge 0$ and strict inequalities hold for at least one pair b_k, β_k and one pair c_k, γ_k. Let $b_0 > 0$, $c_0 > 0$, and $c_0 - \sum_{k=1}^{m} \frac{c_k}{\gamma_k} > 0$. Let φ, u_0, f satisfy the assumptions of Theorem 4.7. Then by Corollary 4.6 all conclusions of Theorem 4.7 if the inequality (4.14) is satisfied.

Proof of Theorem 4.7. Under the assumptions of the theorem the heat flow problem (4.3) is equivalent to the abstract Volterra equation (V) with the kernel b given by equations (b), (ρ), and (4.4), the forcing term given by equations (f), (ρ), and (4.5), and the operator $A = \partial\varphi$ when $\varphi : H \to (-\infty,\infty]$ is the proper, convex, l.s.c. function defined in Example 3.6 (or Example 3.7 in more than one space dimension). To establish the existence and uniqueness of a strong solution of (V) (equivalent to (4.3)), we apply Proposition 1.2. Lemma 4.2 shows that the assumptions of Proposition 1.2 concerning f are satisfied with $f(0,x) = u_0(x) \in H_0^1(0,1) = D(\varphi)$ (see Example 3.6). Example 3.6 also shows that (H_φ) is satisfied. Lemma 4.1 shows that assumptions (H_b) are satisfied. Thus to apply Proposition 1.2 we must still verify that $B' \in BV_{loc}[0,\infty)$. From (4.8) and (4.4) we compute

$$B'(t) = -\frac{\gamma(t)}{b_0} + c_0\rho(t) - (\gamma*\rho)(t) \qquad (0 \le t < \infty) . \tag{4.15}$$

Since β is monotone by hypothesis, the resolvent equation (ρ) and a standard argument (see e.g. Bellman and Cooke [2]) show that ρ is monotone. Finally, since γ is monotone, it follows that $B' \in BV[0,\infty)$. Thus Proposition 1.2 yields the existence and uniqueness of a strong solution u of (V) on $[0,\infty)$ such that $u' \in L^2_{loc}[0,\infty;H)$.

We shall next apply Theorem 3.1. Concerning the kernel b Lemma 4.1 shows that assumptions (H_b) are satisfied with $b_\infty > 0$. Moreover, Corollary 4.4 shows that b

satisfies the frequency domain condition (F) if b_0 and c_0 are chosen as in the statement of Theorem 4.7.

Example 3.6 (or 3.7 in the case of more than one dimension) shows that assumptions (σ), $\varphi(y) \geq 0$ $(y \in H)$, $\lim_{|y| \to \infty} \varphi(y) = +\infty$ hold, and that the inclusion $\partial\varphi(w) \ni 0$ has $w = 0$ as the only solution. Lemma 4.2 shows that assumptions (H_f) are satisfied. Therefore, by Theorem 3.1 the solution u has the properties:

$$\sup_{0 \leq t < \infty} \varphi(u(t)) < \infty, \quad \sup_{0 \leq t < \infty} |u(t)| < \infty, \quad \frac{du}{dt} \in L^2(0,\infty), \quad u(t) \text{ is uniformly continuous on } [0,\infty).$$

$$\lim_{t \to \infty} \varphi(u(t)) = 0, \quad \text{and} \quad u(t) \longrightarrow 0 \quad \text{as} \quad t \to \infty .$$

Example 3.6 also shows that under assumption (σ) the coercivity assumption (2.4) is satisfied for every $T > 0$ with $\varepsilon = p_0\pi^2 > 0$ (or another positive constant in the case of two space dimensions - see Example 3.7). Therefore, by Theorem 3.2 one also has $u \in L^2(0,\infty;H)$ and $u(t) \to 0$ as $t \to \infty$ strongly in H. This completes the proof.

<u>Remark 4.10</u>. Suppose

$$\beta(t) = b_1 e^{-\beta_1 t} \cos\lambda t \qquad (b_1, \beta_1 > 0,\ 0 \leq t < \infty)$$

$$\gamma(t) = c_1 e^{-\gamma_1 t} \cos\omega t \qquad (c_1, \gamma_1 > 0,\ 0 \leq t < \infty) ,$$

and assume that $b_0 > 0$, $c_0 > 0$. Also suppose that σ and h satisfy the assumptions of Theorem 4.7. Although the assumptions concerning β, γ in Theorem 4.7 are not satisfied, one still has by Lemma 4.1 that assumptions (H_b) hold with $b_\infty > 0$ provided

$$(\gamma) \qquad c_0 - \frac{c_1\gamma_1}{\gamma_1^2 + \omega^2} > 0 .$$

Moreover, $B' \in BV_{loc}[0,\infty)$ from (4.15), and the existence and uniqueness of a strong solution of (V) (equivalent to (4.3)) such that $u' \in L^2_{loc}(0,\infty;H)$ follows from Proposition 1.2. Thus to obtain all of the conclusions of Theorem 4.7 we need only verify that the

$$\int_0^T w(t)\,\frac{d}{dt}\,(k*w)(t)\,dt = k(0)\int_0^T w^2(t)\,dt + \int_0^T w(t)(k'*w)(t)\,dt$$

$$= k(0)\int_0^T w^2(t)\,dt + \frac{1}{2}\int_0^T w(t)\int_0^T k'(t-\tau)w(\tau)\,d\tau\,dt$$

$$= k(0)\int_{-\infty}^{\infty} w_T^2(t)\,dt + \frac{1}{2}\int_{-\infty}^{\infty} w_T(t)\int_{-\infty}^{\infty} k'(t-\tau)w_T(\tau)\,d\tau\,dt \ .$$

Letting $\tilde{w}_T(\eta) = \int_{-\infty}^{\infty} e^{i\eta t} w_T(t)\,dt$, $(\eta \in \mathbf{R})$, the Parseval and convolution theorems give

$$\int_0^T w(t)\,\frac{d}{dt}\,(k*w)(t)\,dt = \frac{k(0)}{2\pi}\int_{-\infty}^{\infty} |\tilde{w}_T(\eta)|^2\,d\eta + \frac{1}{4\pi}\int_{-\infty}^{\infty} |\tilde{w}_T(\eta)|^2\,\tilde{k}'(\eta)\,d\eta \ .$$

But $\tilde{k}'(\eta) = 2\mathrm{Re}\ \hat{k}'(i\eta)$, where $\hat{\ }$ is the Laplace transform, and $\mathrm{Re}\ \hat{k}'(i\eta) = \mathrm{Re}[i\eta\hat{k}(i\eta) - k(0)]$. Therefore,

$$\int_0^T w(t)\,\frac{d}{dt}\,(k*w)(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{w}_T(\eta)|^2\,\mathrm{Re}[i\eta\hat{k}(i\eta)]\,d\eta \ .$$

Now an easy calculation from equation (k) yields

$$\mathrm{Re}[i\eta\hat{k}(i\eta)] = \mathrm{Re}\,\frac{1}{\hat{b}(i\eta)} = \frac{\mathrm{Re}\ \hat{B}(i\eta)}{(\mathrm{Re}\ \hat{B}(i\eta))^2 + \left(\mathrm{Im}\ \hat{B}(i\eta) - \frac{b_\infty}{\eta}\right)^2}$$

$$= \frac{\eta^2\,\mathrm{Re}\ \hat{B}(i\eta)}{\eta^2(\mathrm{Re}\ \hat{B}(i\eta))^2 + (\mathrm{Im}\ \hat{B}(i\eta)\eta - b_\infty)^2} \ge 0 \qquad (-\infty < \eta < \infty) \ ,$$

where the last inequality follows from the assumption that B is a kernel of positive type on $[0,\infty)$ (which is equivalent to $\mathrm{Re}\ \hat{B}(i\eta) \ge 0$ [14, Thm. 2]; note that it is impossible to bound $\mathrm{Re}[i\eta\hat{k}(i\eta)]$ away from zero, even if B is strongly positive on $[0,\infty)$).

Appendix 1

Proof of Lemma 2.2: (a) Consider the resolvent equation (k) of B'. Since $B' \in L^1(0,\infty)$ Paley-Wiener theorem [16] yields that $k \in L^1(0,\infty)$ if and only if

$$P(z) = b(0) + \hat{B}'(z) = b_\infty + z\hat{B}(z) \neq 0 \qquad (\mathrm{Re}\ z \geq 0) .$$

With $z = x + iy$

$$\mathrm{Re}\ P(z) = b_\infty + x\ \mathrm{Re}\ \hat{B}(z) - y\ \mathrm{Im}\ \hat{B}(z) \qquad (x \geq 0)$$

$$\mathrm{Im}\ P(z) = x\ \mathrm{Im}\ \hat{B}(z) + y\ \mathrm{Re}\ \hat{B}(z) \qquad (x \geq 0) .$$

Since $P(z)$ is analytic in $\mathrm{Re}\ z > 0$ and continuous in $\mathrm{Re}\ z \geq 0$, $\mathrm{Re}\ P(z)$ and $\mathrm{Im}\ P(z)$ are harmonic for $x \geq 0$. Hence by the maximum principle for harmonic functions, $P(z) \neq 0$ for $x \geq 0$ if either $\mathrm{Re}\ P(iy) = b_\infty - y\ \mathrm{Im}\ \hat{B}(iy)$, or $\mathrm{Im}\ P(iy) = y\ \mathrm{Re}\ \hat{B}(iy)$ are different from zero for $-\infty < y < \infty$. But by the frequency domain condition (F) $\mathrm{Re}\ P(iy) > 0$ for $-\infty < y < \infty$, and thus $k \in L^1(0,\infty)$.

(b) Since $B' \in L^1(0,\infty) \cap L^2(0,\infty)$ and $k \in L^1(0,\infty)$, one has $B' * k = k * B' \in L^2(0,\infty)$, and the result $k \in L^2(0,\infty)$ follows by inspection of equation (k). If also $B'' \in L^1(0,\infty)$, then $B' \in C[0,\infty)$ (so that $|B'(0)| < \infty$) and we may differentiate (k) to obtain

$$b(0)k'(t) + B'(0)k(t) + (B''*k)(t) = -\frac{B''(t)}{b(0)} \qquad (0 \leq t < \infty) ,$$

and clearly $k' \in L^1(0,\infty)$.

(c) If, as is the case here $k' \in L^1(0,\infty)$, the energy inequality in (c) is derived by the following simple argument (see the method of [14, Theorem 1]). Extend k' evenly for $t < 0$, and let

$$w_T(t) = \begin{cases} w(t) & \text{if } t \in [0,T] \\ 0 & \text{otherwise} . \end{cases}$$

Then

frequency domain assumption (F) holds in order to apply Theorems 3.1 and 3.2. By Lemma 4. it suffices to find sufficient conditions on the constants b_0, c_0, b_1, β_1, λ, c_1, γ_1, ω so that (4.12) holds. An elementary, but tedious calculation shows that $\delta > 0$ such that (4.12) holds exists in the case $\gamma_1^2 > \omega^2$, $\beta_1^2 > \omega^2$, provided assumption (γ) above holds, and provided $b_0 > 0$ is chosen sufficiently large.

While no claim is made here that the above functions β and γ represent physically plausible relaxation functions, it is of some interest that the theory can still be applied. In this connection it may also be noted that here the function

$$C(t) = c_0 - \int_0^t \gamma(\tau)d\tau = c_0 - \frac{c_1\gamma_1}{\gamma_1^2 + \omega^2}(1 - e^{-\gamma_1 t}\cos \omega t) - \frac{c_1\omega}{\gamma_1^2 + \omega^2} e^{-\gamma_1 t}\sin \omega t \ .$$

In a genuinely physical problem as motivated above one would need to require $C(t) > 0$ $(0 \leq t < \infty)$, as well as assumption (γ). However, in the application of the theory the physical requirement $C(t) > 0$ $(0 \leq t < \infty)$ is not used and indeed, for example,

$$C\left(\frac{\pi}{2}\right) = c_0 - \frac{c_1\gamma_1}{\gamma_1^2 + \frac{\pi^2}{4}} - \frac{\frac{\pi}{2}\, c_1}{\gamma_1^2 + \frac{\pi^2}{4}} e^{-\gamma_1 \frac{\pi}{2}}$$

could be negative, even though $C(\infty) = c_0 - \dfrac{c_1\gamma_1}{\gamma_1^2 + \frac{\pi^2}{4}} > 0$ holds.

(d) Multiply equation (k) by $\sqrt{t}$:

$$b(0)\sqrt{t}\ k(t) + \sqrt{t}\ (B'*k)(t) = -\frac{\sqrt{t}\ B'(t)}{b(0)} \qquad (0 \leq t < \infty) .$$

An elementary calculation involving $\sqrt{t}(B'*k)$ shows that $\sqrt{t}\ k$ satisfies

$$b(0)\sqrt{t}\ k(t) + \int_0^t B'(t-\tau)\sqrt{\tau}\ k(\tau)d\tau = -\frac{\sqrt{t}\ B'(t)}{b(0)} - \int_0^t (\sqrt{t} - \sqrt{\tau})B'(t-\tau)k(\tau)d\tau$$

$$(0 \leq t < \infty) .$$

Since $\sqrt{t} - \sqrt{\tau} \leq \sqrt{t-\tau}$ for $0 \leq \tau \leq t$, and since also $\sqrt{t}\ B' \in L^1(0,\infty)$ by assumption and $k \in L^1(0,\infty)$ by (a), the integral on the right side of the last equation defines a function in $L^1(0,\infty)$. Then $\sqrt{t}\ k \in L^1(0,\infty)$ by the argument of part (a). The additional assumptions and elementary estimates also yield $\sqrt{t}\ k \in L^2(0,\infty)$.

Finally, differentiating the equation (k) as in part (b), multiplying the resulting equation by $\sqrt{t}$, and using elementary estimates yields $\sqrt{t}\ k' \in L^1(0,\infty)$. This completes the proof of Lemma 2.2.

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We study the boundedness and asymptotic properties of solutions as $t \to \infty$ of the nonlinear Volterra equation

(V) $$u(t) + (b*Au)(t) \ni f(t) \qquad (0 \le t < \infty),$$

where $b : [0,\infty) \to \mathbb{R}$ is a given kernel, A is a maximal monotone (possibly multi-valued) operator on a real Hilbert space H, $*$ is the convolution, and

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$f : [0,\infty) \to H$ is a given function. The special case $A = \partial\varphi$, where $\varphi : H \to (-\infty,+\infty]$ is a proper, convex, l.s.c. function is also considered. The problem of existence and uniqueness of solutions has been studied previously. Two principal types of results are derived; sufficient conditions are obtained on the kernel b, the operator A, and the forcing term f such that either (i) $u \in L^{\infty}(0,\infty;H)$ and $u \to 0$ as $t \to \infty$ strongly in H, or (ii) $u \in L^{\infty}(0,\infty;H)$ and $u \to u_{\infty}$ as $t \to \infty$ strongly in H, where u_{∞} is the unique solution of an appropriate limiting equation associated with (V). The results are established by obtaining a priori estimates by combining energy methods with frequency domain techniques. The results are natural generalizations for the abstract equation (V) of comparable known results for the scalar equation in which A is a real function. Of several applications discussed the principal one is an analysis of the asymptotic behaviour of solutions of the physically interesting problem of heat flow in a material with memory.